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REPEATED GAMES WITH FINITE AUTOMATA

by

Elchanan Ben-Porath

Technical Report No. 515

August 1987

A REPORT OF THE
CENTER FOR RESEARCH ON ORGANIZATION EFFICIENCY
STANFORD UNIVERSITY
CONTRACT N00014-86-K-0216
United States Office of Naval Research

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REPEATED GAMES WITH FINITE AUTOMATA

by

Elchanan Ben-Porath

1. Introduction

~~In~~ this paper ~~we~~ examines the set of equilibrium payoffs in a repeated game when there are bounds on the complexity of the strategies that players may select.

The interest in putting such bounds comes from the limited computational ability[1] of humans and devices used by humans, (see Simon [1957, 1972]). For example most of the strategies in a repeated game cannot be implemented by any computer.

It is important to distinguish between the complexity of a strategy and the complexity of the process of selecting a strategy. ~~We will~~ not deal with the selection process directly. ~~We will~~ assume that the limitation of a player is such that he can consider all the strategies below a certain level of complexity. A possible interpretation is that the players' abilities are unbounded but they use bounded devices to implement their strategies (secretaries, or computers).

* This work was partially supported by Office of Naval Research Grant N000-14-86-K-0216 at the Institute for Mathematical Studies in the Social Sciences, Stanford University, Stanford, California. This work is based on my M.S. thesis. My advisor, Professor Abraham Neyman introduced me to the topic and guided me. I owe him many thanks.

We use the notion of a finite automaton to define a complexity measure on the strategies. A finite automaton is a machine that contains a finite number of states. One of these states is the initial state. The machine has an action function and a transition function. The action function determines the one-shot game strategy that is played at each state. The transition function specifies the next state as a function of the current state and the current action of the other players. An automaton induces a strategy as follows: The state at the first stage is the initial state. The state and the actions of the other players at stage t determine the state at stage $t + 1$ (by the transition function). The state determines the one-shot game strategy (by the action function). The size of an automaton is the number of states it has. The complexity of a strategy is defined as the size of the minimal automaton that can implement it. The main result is that in a zero-sum game, when the size of the automata of both players go together to infinity the sequence of values converges to the value of the one-shot game. This is true even if the size of the automata of one player is a polynomial of the size of the automata of the other player. This means that a player can gain something from being able to use strategies which are more complicated than the strategies of his opponent, only if his strategies are <u>much</u> more complicated. The result for the zero-sum games gives an estimation for the general case.

The idea of using a finite automaton in order to distinguish between simple and complicated strategies was proposed by Aumann [1981]. In two recent works Neyman [1985] and Rubinstein [1985] used the notion of a

finite automaton to explore the issue of strategic complexity in repeated games. Neyman shows that in the finitely repeated prisoners dilemma, putting a bound on the complexity of the strategies can induce cooperation. Rubinstein in the infinite version, does not restrict the players to a certain level of complexity, instead players seek to minimize the complexity of their strategies provided they do not decrease their payoff. This eliminates many of the equilibrium payoffs.

The model in the present study was proposed to me by Neyman.

2. The Model

Let G be a zero-sum game $G = (S^1, S^2, r)$; where S^i is a finite set of actions for players i and $r\colon S^1 \times S^2 \to R$ is the payoff function of player I. Let $V(G)$ denote the value of the game and let maxmin (G) and minmax (G) denote $\max_{s^1 \in S^1} \min_{s^2 \in S^2} r(s^1, s^2)$ and $\min_{x^2 \in S^2} \max_{s^1 \in S^1} r(s^1, s^2)$ respectively.

An automaton A^i for player i is a four-tuple $\langle M^i, \bar{q}^i, f^i, g^i \rangle$ where M^i is a set, $\bar{q}^i \in M^i$ $f^i\colon M^i \to S^i$ and $g\colon M^i \times S^j \to M^i$ $(j \neq i)$. M^i is the set of states of the automaton, $\bar{q}^i$ is the initial state, $f^i(q^i)$ is the action the player chooses when the automaton is at state q^i and g^i is the transition function, if the automaton is at state q^i and the other player chooses s^j the next state is $g^i(q^i, s^j)$. An automaton is finite, if the sets of states is finite. We will consider only finite automata. The size of an automaton is the number of the states. An automaton of player i

induces a (pure) strategy in the repeated game as follows: The action at stage t is $f^i(q_t^i)$ where q_t^i is the state of the automaton at stage t. The sequence of states is determined inductively by $q_1^i = \bar{q}^i$ $q_{t+1} = g^i(q_t^i, s_t^j)$ where s_t^j is the action of the other player at stage t. For example consider the game:

	L	R
T	1	-1
B	-1	1

The strategy of player I which begins with T, continues with it as long as player I chooses L and plays B forever if player II plays R, is induced by the automaton $A = \langle M, q, f, g\rangle$; where

$$M = \{1,2\}; \quad q = 1; \quad f(1) = T \quad f(2) = B$$

$$g(1,L) = 1 \quad g(1,R) = 2 \quad g(2,L) = g(2,R) = 2.$$

Given that the automata of the players are A^1 and A^2 the corresponding strategies in the repeated game determine a sequence of actions and payoffs. Denote by $R_t(A^1,A^2)$ the payoff at stage t. The payoff when player I chooses A^1 and player II chooses A^2 is defined to be the limit of the means[2/]:

$$(1) \qquad R(A^1,A^2) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} R_t(A^1,A^2)$$

We are interested in the value of the game where each player is restricted to strategies that can be implemented by an automaton of a given size.

Formally, define:

$$A_n^1 = \{A^1 | A^1 \text{ is an automaton of size } n \text{ for player I}\}$$

$$A_m^2 = \{A^2 | A^2 \text{ is an automaton of size } m \text{ for player II}\}$$

$$G_{n,m} = (A_n^1, A_m^2, R).$$

Thus $G_{n,m}$ is the game induced by restricting player I and player II to strategies that can be implemented by automata of size n and m respectively. Note that $G_{n,m}$ is also a zero-sum game.

I assume without loss of generality that the set of states of an automaton of size n is $\{1,\ldots,n\}$. With this identification A_n^1 and A_m^2 are finite. Let $k = |S^1|$ and $h = |S^2|$. Then $|A_n^1| = n \cdot k^n \cdot n^{n\times h}$ and $|A_m^2| = m \cdot h^m \cdot m^{m\times k}$.

3. The Asymptotic Behavior of $V(G_{n,m})$

The main result in this section is that if $p(n)$ is a polynomial $\lim_{n\to\infty} V(G_{n,p(n)}) = V(G)$.

First note that player I can get at least the maxmin (G) by playing constantly the maxmin action. An automaton of size one can implement this strategy and of course any larger automaton can do it. Similarly player II can get minmax (G). Thus the following inequalities hold

$$\text{maxmin } (G) \leq V(G_{n,m}) \leq \text{minmax } (G).$$

The first result states that for any number n there exists a larger number m such that $V(G_{n,m}) = \text{maxmin } G$.

First we need a definition and a lemma.

Definition: (a) A^i is a partial automaton if the transition function g is a defined on a subset of $M^i \times S^j$.

(b) For two partial automata A and A', A' is an extension of A if its transition function is an extension of the transition function of A. If A' is an automaton it will be called a completion of A.

Lemma (1.1): For every $A^1 \in A_n^1$ there exists a partial automaton, A^2 of size n such that for any completion $\bar{A}^2$ of A^2

(2) $$R(A^1,\bar{A}^2) \leq \text{maxmin } G$$

Proof: Define

$$h:\ S^1 \to S^2$$

$$h(s^1) = \text{argmin}_{s^2 \in S^2} r(s^1,s^2)$$

$H(s^1)$ is the best reaction for player II to the action s^1 of player I. Denote the different states of A^1 by $\{1,\ldots n\}$ and assume that the initial state is 1. Define a partial automaton A^2 as follows:

$$M^2 = \{1,\ldots,n\} \quad \bar{q}^2 = 1 \quad f^2(i) = h(f^1(i)) \quad 1 \leq i \leq n$$

g^2 is a partial function which satisfies

$$g^2(i,f^1(i)) = g^1(i,f^2(i)) \qquad i \leq i \leq n$$

It is easy to see that g^2 can be defined in such a way and that every completion of A^2 satisfies (2).

Theorem 1: If $L(n) \geq n^2 \; |A_n^1| = n^3 \cdot k^n \cdot n^{n \times h}$ there exists an automaton $A^2 \in A^2_{L(n)}$ such that for every $A^1 \in A_n^1$ $R(A^1,A^2) \leq$ maxmin (G).

We will construct A^2. Roughly A^2 operates as follows: it identifies the automaton A^1 and then uses a subautomaton from the type described in lemma 1.1.

Proof: Let A_ℓ^1 $1 \leq \ell \leq |A_n^1|$ be an automata of player I. With every A_ℓ^1 associate n identical partial automata $A^2_{\ell,1},\ldots,A^2_{\ell,n}$ from the type defined in lemma 1.1

The set of states of any two different partial automata $A^2_{i,q} \in A^2_{j,r}$ ($i \neq j$ or $q \neq r$) are disjoint. Together they form a partial automaton with $n^2|A_n^1|$ states. Define the initial state to be the initial state of $A^2_{1,1}$. Denote this partial automaton by $\bar{A}^2$. We will extend the automaton inductively so that the partial automaton at stage p, A_p^2 will satisfy

$$R(A_s^1,A_p^2) \leq \text{maxmin } (G)$$

$$\text{for } 1 \leq s \leq p$$

Define

$$A_1^2 = \bar{A}^2$$

Let $p > 1$ and play A_{p+1}^1 and A_p^2. Let t be the first time that A_{p+1}^1 "reveals his identity" i.e. the action played by A_{p+1}^1 - s_t^1 - is such that the sequence $(s_1^1,\ldots,s_t^1)$ is different from the action sequences of the automata $A_1^1,\ldots,A_p^1$. (If such a t doesn't exist define $A_{p+1}^2 = A_p^2$.)

Denote:

q_t^2 - the state of A_p^2 at stage t.

q_{t+1}^1 - the state of A_{p+1}^1 at stage $t + 1$ (q_{t+1}^1 is determined at stage t.)

$ai(p + 1)$ - the index of the automaton which is identical to A_{p+1}^1 but has q_{t+1}^1 as an initial state.

$ci(p + 1)$ - the index of the first copy of the automata $A_{ai(p+1),1}^2,\ldots,A_{ai(p+1),n}^2$ that hasn't been used yet.

(Formally $ci(p + 1) = \min \{j|$ there doesn't exist $1 \leq r \leq p + 1$ such that

$$ai(r) = ai(p + 1) \text{ and } ci(r) = j\}.)$$

The inductive step is to set $g^2(q_t^2,s_t^1)$ to the initial state of $A^2_{ai(p+1),ci(p+1)}$.

It is easy to see that this definition is an extension and (1) is satisfied. Finally define

$$A^2 = A^2_{|A_n^1|} .$$

<u>Theorem 2</u>: Let $Q(n)$ be a function which satisfies $Q(n) \geq n$ and $\lim_{n\to\infty} \frac{\ell n[Q(n)]}{n} = 0$ then $\lim_{n\to\infty} V(G_{n,Q(n)}) = V(G)$.

<u>Proof</u>: For each n we will define a mixed strategy P^n for player I (i.e. a probability distribution on A_n^1), such that for every $A^2 \in A^2_{Q(n)}$ $\sum_{A^1 \in A_n^1} P^n(A^1) \cdot R(A^1,A^2) \geq V(G) - \varepsilon_n$ where $\lim_{n\to\infty} \varepsilon_n = 0$.

Assume $S^1 = \{1,\ldots,k\}$ and $S^2 = \{1,\ldots,h\}$. Define $\Omega_n = \{S^1\}^n$. Let $w \in \Omega_n$ $w = (w_1,\ldots w_n)$. Denote by A_w^1 $(A_w^1 \in A_n^1)$ the following automaton:

$$M^1 = \{1,\ldots,n\}, \quad \bar{q}^1 = 1, \quad f^1(i) = w_i, \quad g(i,s_j^2) = \begin{cases} i+1 & i < n \\ 1 & i = n \end{cases}$$

Let $(P_1,\ldots,P_k)$ be a mixed strategy for player I in the game G, such that $\sum_{i=1}^k P_i \cdot r(i,j) \geq V(G)$ for every $j \in S^2$. Define a probability measure on $(\Omega_n, 2^{\Omega_n})$ by

$$\mu_n(w) = \Pi_{i=1}^n P_{w_i} .$$

The strategy P^n of player I in the game $G_{n,Q(n)}$ is defined by

$$P^n(A^1_w) = \mu_n(w).$$

We have to show that,

(1) For every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for every $n > N(\varepsilon)$ and for every $A^2 \in A^2_{Q(n)}$ (1') is satisfied

$$(1') \qquad \sum_{w \in \Omega_n} \mu_n(w) R(A^1_w, A^2) \geq V(G) - \varepsilon.$$

It suffices to show that,

(2) for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for every $n > N(\varepsilon)$ $A^2 \in A^2_{Q(n)}$ and $c \in N$ (2') is satisfied.

$$(2') \qquad \sum_{w \in \Omega_n} \mu_n(w) \cdot \frac{1}{n} \sum_{t=c\cdot n+1}^{(c+1)n} R_t(A^1_w, A^2) \geq V(G) - \varepsilon$$

Let $A^2 \in A^2_{Q(n)}$ and assume $M^2 = \{1,\ldots,Q(n)\}$. Let $A^2(p)$ $p = 1,\ldots,Q(n)$ denote the automaton which is identical to A^2 but with initial state p. Let $q(A^2,w,c)$ denote the state of the automaton A^2 in the stage $c \cdot n + 1$, when player I chooses A_w.

Finally define:

$$(3)\qquad X_{A^2,c}(w) \overset{\text{def}}{=} \frac{1}{n}\sum_{t=c\cdot n+1}^{(c+1)\cdot n} R_t(A_w^1,A^2) = \frac{1}{n}\sum_{t=1}^{n} R_t(A_w^1,A^2(q(A^2,w,c)))$$

Note that the left expression in (2') is the expectation of $X_{A_2,c}$. To estimate the expectation and compare it to $V(G)$, we will estimate

$$(4)\qquad \mu_n\{w:\ X_{A^2,c}(w) < V(G) - \varepsilon\}.$$

The important point in the proof is that the state of A^2 in stage ℓ, $1 \le \ell \le n$ is determined by the first $\ell - 1$ actions of player I. Thus, with every action $j = 1,\dots,n$ of player II we can associate a sequence of random variables $(f_{j,\ell})_{\ell=1,\dots,n}$

$$f_{j,\ell}(w) \overset{\text{def}}{=} \begin{cases} 1 & \text{the action of } A^2 \text{ at stage } \ell \\ & \text{when it plays against } A_w \text{ is } j \\ & \\ 0 & \text{otherwise} \end{cases}$$

and $f_{j,\ell}$ are measurable w.r.t. the algebra which is generated by the first $\ell - 1$ elements.

We now need two lemmas.

Given an automaton A^2, call a sequence $w \in \Omega_n$ "nice", if the number of times the actions (i,j) are played divided by the number of times j is played, is 'close' to P_i. Lemma 2.1 says that 'almost' all the sequences are "nice". Lemma 2.2 states that on a "nice" sequence the average payoff is 'close' to $V(G)$.

Lemma 2.1: Let $A = \{a_1,\ldots,a_k\}$ be a finite set. Let $(A,2^A,P)$ be a probability space and let $(\Omega,2^\Omega,\mu)$ be the product space $(A,2^A,P)^n$ i.e. $\Omega = A^n$ and for every $w = (w_1,\ldots,w_n) \in A^n$ $\mu(w) = \Pi_{i=1}^n P(w_i)$. Let H_ℓ be the partial algebra of 2^Ω which is generated by $w_1,\ldots,w_{\ell-1}$ $H_1 = \{\phi,\Omega\}$. Let $\{f_{j,\ell}\}_{\ell=1,\ldots,n}^{j=1,\ldots,h}$ be a set of random variables which are adapted to $(H_\ell)_{\ell=1,\ldots,n}$ (i.e., $f_{j,\ell}$ is measurable w.r.t. (Ω,H_ℓ)) and with values in the set $\{0,1\}$. Define

$$S_\varepsilon = \left\{ w: \left|\frac{1}{n}\sum_{\ell=1}^{n} I(w_\ell = a_i)(w)\cdot f_{j,\ell}(w) - \frac{1}{n}\sum_{\ell=1}^{n} P(a_i)\cdot f_{j,\ell}(w)\right| > \varepsilon \right\}$$

$$\text{for some } 1 \leq j \leq h \quad 1 \leq i \leq k$$

There exists $b > 0$ such that for every $b > \varepsilon > 0$

$$\mu(S_\varepsilon) \leq 2\cdot k\cdot h\cdot e^{-\frac{\varepsilon^2 n}{4}}.$$

In lemma 2.2 Ω_n and $f_{j,\ell}$ refer to $\{S^1\}^n$ and the indicator functions of the actions respectively, as defined before Lemma 2.1. let $W(G)$ denote $\max_{s^1\in S^2,\ s^2\in S^2}|r(s^1,s^2)|$.

Lemma 2.2: Define

$$S_{\varepsilon,A^2} = \left\{ \begin{array}{l} w:\ w\in\Omega_n,\ \left|\frac{1}{n}\sum_{\ell=1}^{n} I(w_\ell = i)(w)\cdot f_{j,\ell}(w) - \frac{1}{n}\sum_{\ell=1}^{n} P_i\cdot f_{j,\ell}(w)\right| > \varepsilon \\ \qquad\qquad \text{for some } 1 \leq j \leq h \quad 1 \leq i \leq k \end{array} \right\}$$

If $w \in \Omega_n - S_{\varepsilon,A^2}$ then $V(G) - \frac{1}{n}\sum_{t=1}^{n} R_t(A_w^1, A^2) < W(G) \cdot k \cdot h \cdot \varepsilon$.

The proofs of the lemmas are in the Appendix.

We can now evaluate (4)

$$\{w:\ X_{A^2,c}(w) < V(G) - \varepsilon\} \subseteq \bigcup_{p=1}^{Q(n)} \{w:\ X_{A^2(p),0}(w) < V(G) - \varepsilon\}$$

Let $\eta = \frac{\varepsilon}{W(G) \cdot k \cdot h}$. Note that by lemma 2.2 if $w \in \Omega_n - S_{\eta,A^2(p)}$ then $X_{A^2(p),0}(w) > V(G) - \varepsilon$. This and lemma 2.1 imply that for every $p = 1,\ldots,Q(n)$

$$\mu_n\{w:\ X_{A^2(p),0}(w) < V(G) - \varepsilon\} \leq \mu_n(S_{\eta,A^2(p)}) \leq 2 \cdot k \cdot h \cdot e^{\frac{-\eta^2 \cdot n}{4}}.$$

From this

$$\mu_n\{w:\ X_{A^2,c}(w) < V(G) - \varepsilon\} \leq \sum_{p=1}^{Q(n)} \mu_n\{w:\ X_{A^2(p),0}(w) < V(G) - \varepsilon\}$$

$$\leq Q(n) \cdot 2 \cdot k \cdot h \cdot e^{\frac{-\eta^2 \cdot n}{4}}.$$

Since $\lim_{n\to\infty} \frac{1}{n} \ln Q(n) = 0$, for every $\eta > 0$

$\lim_{n\to\infty} Q(n) \cdot 2 \cdot k \cdot h \cdot e^{\frac{-\eta^2 \cdot n}{4}} = 0$. Hence for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for any $n \geq N(\varepsilon)$, $A_2 \in A^2_{Q(n)}$ and $c \in N$, (5) is satisfied.

$$(5) \qquad \mu_n\{w | X_{A^2,c}(w) < V(G) - \varepsilon\} < \varepsilon$$

Since $X_{A^2,c}$ are uniformly bounded, i.e. $(\forall c, \forall n, \forall A^2 \in A^2_{Q(n)}\ |X_{A^2,c}(w)| \leq W(G))$ we get (2).

We have proved $\underline{\lim}_{n\to\infty} V(G_{n,Q(n)}) \geq V(G)$. Since $Q(n) \geq n$ a similar argument shows that $\overline{\lim}_{n\to\infty} V(G_{n,Q(n)}) \leq V(G)$. These inequalities imply the result.

4. Non Zero-Sum Games

Theorem 2 gives an estimation for the asymptotic behavior of the set of equilibrium payoffs in an m-person game. Let G be an m-person game. An automaton for player i is the same as was previously defined, except that the transition function is defined on $M^i \times S^{-i}$ where S^{-i} is the set of action tuples of the other players. Let $S(z_1,\ldots,z_m)$ denote the set of equilibrium payoffs in the repeated game when player i is restricted to strategies that can be implemented by automata of size z_i. Let $\underline{S}$ denote the set of individual rational payoffs in mixed strategies. Let co G denote the convex hull of the vectors in the payoff matrix of G. Let $y = (y_1,\ldots,y_m)$ denote the vector of maximal payoffs that each player can get regardless of what the other players do (in mixed strategies). Define

$$\overline{S} = \{x: \ x \in \text{co } G \quad x \geq y\}.$$

<u>Theorem 3</u>: Let $Q_2(n),\ldots,Q_m(n)$ be functions such that $Q_i(n) \geq n$ and $\lim_{n\to\infty} \frac{\ln[Q_i(n)]}{n} = 0$. Then

$$\underline{S} \subseteq \underline{\lim}_{n\to\infty} S(n,Q_2(n),\ldots,Q_m(n)) \subseteq \overline{\lim}_{n\to\infty}(S(n,Q_2(n),\ldots,Q_m(n)) \subseteq \overline{S}.$$

It is easy to see that every rational convex combination of the payoff vectors in G can be implemented by automata which are large enough. Player i can get y_i by using the strategy that was defined in Theorem 2. The other players can bring player i down to his individual rational payoff by using the same type of strategies. The proofs of the last two claims are similar to the proof of Theorem 2. I omit the details.

Corollary: Let G be a 2-person game and let $Q(n)$ be a function that satifies $Q(n) \geq n$ and $\lim_{n\to\infty} \frac{\ln[Q(n)]}{n} = 0$. Then

$$\lim_{n\to\infty} S(n,Q(n)) = \underline{S} = \overline{S}.$$

5. Conclusion

So far we have considered a specific measure of complexity. However a version of Theorem 2 is true for a large class of measures.

Definition: A function $g\colon N \to N$ is log - polynomial bounded (henceforth l.p.b.) if there exists a polynomial p such that $\ln(g(n)) \leq p(\ln(n))$.

Note that every polynomial is a l.p.b.

Consider a complexity measure as a function from the set of strategies to the natural numbers.

Definition: Two measures of complexity C_1, C_2 are log - polynomial equivalent if there exist l.p.b. functions $g_1(n)$ $g_2(n) \geq n$ such that

$$C_1^{-1}\{x: \ x \leq n\} \subseteq C_2^{-1} \ \{x: \ x \leq g_1(n)\}$$

$$C_2^{-1}\{x: \ x \leq n\} \subseteq C_1^{-1} \ \{x: \ x \leq g_2(n)\}$$

Let C be a complexity measure. $G^C_{n,m}$ denotes a game that is similar to $G_{n,m}$ except that the complexity of the strategies is measured by C. (So player I, for example, is restricted to strategies that according to C have a complexity that is less than n.)

<u>Theorem 4</u>: Let C be a complexity measure that is log-polynomial equivalent to the automata measure and let $Q(n)$ be a l.p.b. function that satisfies $Q(n) \geq n$. Then

$$\lim_{n\to\infty} V\left(G^C_{n,Q(n)}\right) = V(G).$$

<u>Proof</u>: There exists a l.p.b. function g_1 such that player II is restricted to strategies that can be implemented by an automaton of size $g_1(Q(n))$. Since g_1 and Q are l.p.b. there exists polynomials p_1 p_2
such that $\ln[g_1(Q(n))] \leq p_1[\ln(Q(n))]$ and $\ln[Q(n)] \leq p_2[\ln(n)]$.
Putting this together gives:

(1) $$\ln[g_1(Q(n))] \leq p_1[p_2(\ln(n))].$$

There exists a l.p.b. function g_2 such that for every $x \in N$ that satisfies $g_2(x) \leq n$ player I can use any strategy that can be implemented by an automaton of size x. Let m be the largest number such that $m \leq n$ and $g_2(m) \leq n$. We have $g_2(m) + 1 > n$. Since g_2 is

l.p.b. there exists a polynomial p_3 such that $\ln[g_2(m) + 1] \leq p_3[\ln(m)]$. Putting together with (1) we get:

$$\ln[g_1(Q(n))] \leq p_1(p_2(p_3(\ln(m)))).$$

A composition of polynomials is a polynomial hence

$$\lim_{m\to\infty} \frac{p_1[p_2[p_3[\ln(m)]]]}{m} = 0.$$

Theorem 2 implies $\underline{\lim}_{n\to\infty} V(G^C_{n,Q(n)}) \geq V(G)$.

Since $Q(n) \geq n$ a similar calculation gives $\overline{\lim}_{n\to\infty} V(G^C_{n,Q(n)}) \leq V(G)$.

The last two inequalities imply the result.

APPENDIX

Proof of Lemma 2.1: We will show if f_ℓ is measurable w.r.t. H_ℓ then,

$$\forall a \in A \; \mu\{w: \left| \frac{1}{n} \sum_{\ell=1}^{n} I(w_\ell = a)(w) \cdot f_\ell(w) - \frac{1}{n} \sum_{\ell=1}^{n} P(a) \cdot f_\ell(w) \right| > \varepsilon\} \leq 2 \cdot e^{\frac{-\varepsilon^2 n}{4}}$$

It is easy to see that this implies the lemma.

Proof: Define

$$Z_\ell = I(w_\ell = a) - P(a)$$

$$Y_\ell = Z_\ell \cdot f_\ell$$

It suffices to show that there exists $\lambda > 0$ such that

(1) $$\mu\{w: \left|\lambda \cdot \sum_{\ell=1}^{n} Y_\ell\right| > \lambda \cdot \varepsilon \cdot n\} \leq 2 \cdot e^{\frac{-\varepsilon^2 n}{4}}$$

We will show that there exists $\lambda > 0$ which satisfies:

(2) $$\mu\{w: \lambda \cdot \sum_{\ell=1}^{n} Y_\ell > \lambda \cdot \varepsilon \cdot n\} \leq e^{\frac{-\varepsilon^2 n}{4}}$$

(3) $$\mu\{w: \lambda \cdot \sum_{\ell=1}^{n} Y_\ell < -\lambda \cdot \varepsilon \cdot n\} \; e^{\frac{-\varepsilon^2 n}{2}}$$

This implies (1). We will show (2). (3) can be proved in a similar way.

Z_ℓ are independent w.r.t. H_ℓ, hence $E(\exp(\lambda \cdot Z_\ell) \mid H_\ell) =$ $E(\exp(\lambda \cdot Z_\ell))$. Exp is a convex function therefore:

(4) $$1 = \exp(E(\lambda \cdot Z_\ell)) \leq E(\exp(\lambda \cdot Z_\ell))$$

By the Taylor expansion

$$\exp(\lambda \cdot Z_\ell) = 1 + \lambda \cdot Z_\ell + \frac{\lambda^2 \cdot Z_\ell^2}{2} + R(\lambda \cdot Z_\ell)$$

where

$$\lim_{\lambda} \to 0 \quad \frac{R(\lambda \cdot Z_\ell)}{\lambda^2 \cdot Z_\ell^2} = 0$$

Hence there exists $d > 0$ such that if $d > \lambda \geq 0$

$$\exp(\lambda \cdot Z_\ell) \leq 1 + \lambda \cdot Z_\ell + \lambda^2 \cdot Z_\ell^2$$

which implies

(5) $$E(\exp(\lambda \cdot Z_\ell)) \leq 1 + \lambda \cdot E(Z_\ell) + \lambda^2 \cdot E(Z_\ell^2) \leq 1 + \lambda^2$$

$Z_1,\ldots,Z_n$ are i.i.d. from (4) and (5) we have

$$1 \leq E(\exp(\lambda \cdot \sum_{\ell=1}^{n} Z_\ell)) \leq (1 + \lambda^2)^n$$

<u>Claim</u>: $\forall \ell \quad 1 \leq \ell \leq n$

(6) $$E(\exp(\lambda \cdot Y_\ell) \mid H_\ell) \leq E(\exp(\lambda \cdot Z_\ell) \mid H_\ell) = E(\exp(\lambda \cdot Z_\ell)$$

<u>Proof</u>: When $f_\ell = 1$ $Y_\ell = Z_\ell$. When $f_\ell = 0$ $Y_\ell = 0$ and thus the left expression equals one while the right expresson is greater than or equal to one.

<u>Claim</u>: $\forall_j \quad 1 \leq j \leq n$

(7) $$E(\exp(\lambda \cdot \sum_{\ell=1}^{j} Y_\ell)) \leq E(\exp(\lambda \cdot \sum_{\ell=1}^{j} Z_\ell))$$

Proof: By induction. For $j = 1$ the claim follows from (6).

Let $j > 1$

$$E(\exp(\lambda \cdot \sum_{\ell=1}^{j} Y_\ell)) = E(E(\exp(\lambda \cdot \sum_{\ell=1}^{j-1} Y_\ell) \cdot \exp(\lambda \cdot Y_j)|H_j))$$

since $\sum_{\ell=1}^{j-1} Y_\ell$ is measurable w.r.t. H_j

$$= E(\exp(\lambda \cdot \sum_{\ell=1}^{j-1} Y_\ell) \cdot E(\exp(\lambda \cdot Y_j)|H_j))$$

$$\leq E(\exp(\lambda \cdot Z_j)) \cdot E(\exp(\lambda \cdot \sum_{\ell=1}^{j-1} Y_\ell)) \quad \text{by (6)}$$

$$\leq E(\exp(\lambda \cdot Z_j)) \cdot E(\exp(\lambda \cdot \sum_{\ell=1}^{j-1} Z_\ell)) \quad \text{by the induction hypothesis}$$

$$= E(\exp(\lambda \cdot \sum_{\ell=1}^{j} Z_\ell)) \quad \text{since } Z_1,\ldots,Z_j \text{ are i.i.d.}$$

From (7) and (5) we get

$$E(\exp(\lambda \cdot \sum_{\ell=1}^{n} Y_\ell)) \leq (1 + \lambda^2)^n$$

By Chebychev inequality

$$\mu\{w: \lambda \cdot \sum_{\ell=1}^{n} Y_\ell > \lambda \cdot \varepsilon \cdot n\} \cdot \exp(\lambda \cdot \varepsilon \cdot n) \leq (1 + \lambda^2)^n$$

$$\mu\{w: \lambda \cdot \sum_{\ell=1}^{n} Y_\ell > \lambda \cdot \varepsilon \cdot n\} < (1 + \lambda^2)^n \cdot \exp(-\lambda \cdot \varepsilon \cdot n)$$

For $\lambda = \frac{\varepsilon}{2}$ the right expression is less than $e^{\frac{-\varepsilon^2 n}{4}}$ (since $(1+\lambda^2)^n \leq e^{\frac{\varepsilon^2 n}{4}}$), hence for $0 < \varepsilon \leq 2d$ (d is the constant derived from the Taylor expansion of Exp) (2) is satisfied.

Proof of Lemma 2.2: Denote:

$\ell_{ij}(w)$ - the number of times (i,j) was played

$x_j(w)$ - the number of times A^2 played j. $x_j(w) = \sum_{i=1}^{k} \ell_{ij}(w)$.

From the definitions

$$\ell_{ij}(w) = \sum_{\ell=1}^{n} I(w_\ell = i)(w) \cdot f_{j\ell}(w)$$

$$x_j(w) \cdot P_i = \sum_{\ell=1}^{n} P_i \cdot f_{j\ell}(w)$$

Hence if $w \in \Omega_n - S_{\varepsilon,A^2}$ then for every $1 \leq i \leq k$ $1 \leq j \leq h$

$$\left| \frac{\ell_{ij}(w)}{x_j(w)} - P_i \right| \leq \frac{n}{x_j(w)} \cdot \varepsilon$$

From this

$$\left| \frac{x_j(w)}{n} \cdot \left(\sum_{i=1}^{k} \frac{\ell_{ij}(w)}{x_j(w)} \cdot r(i,j) - \sum_{i=1}^{n} P_i \cdot r(i,j) \right) \right|$$

$$\leq \frac{x_j(w)}{n} \cdot \sum_{i=1}^{k} \left| \frac{\ell_{ij}(w)}{x_j(w)} - P_i \right| \cdot |r(i,j)| \leq k \cdot W(G) \cdot \varepsilon$$

For every $1 \leq j \leq h$ $\sum_{i=1}^{k} P_i \cdot r(i,j) \geq V(G)$

Hence summing over j gives the result.

FOOTNOTES

1/ I prefer not to use the term "bounded rationality" here, since it has been used as a name for a large class of limitations. For example partial information of possible actions or of consequences of a given action has also been referred to as "bounded rationality" even when there is no computation that will give additional information.

2/ Since the set of the states of each automaton is finite the automata enter a cycle, i.e. there exists numbers c, $k \leq |M^1|\ |M^2|$ such that for every $t \geq c$ $(q^1_t, q^2_t) = (q^1_{t+k}, q^2_{t+k})$ and so the limit (1) exists.

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